Making Sense of the Standards for Mathematical Practice

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Teacher Leaders Making it Happen: Bringing the Guiding Principles and Standards for Mathematical Practice into Mathematics Classrooms

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The Standards for Mathematical Practice

- 1: Make sense of problems and persevere in solving them.
- 2: Reason abstractly and quantitatively.
- 3: Construct viable arguments and critique the reasoning of others.
- 4: Model with mathematics.
- 5: Use appropriate tools strategically.
- 6: Attend to precision.
- 7: Look for and make use of structure.
- 8: Look for and express regularity in repeated reasoning.

The Standards for Mathematical Practice

“varieties of expertise that mathematics educators at all levels should seek to develop in their students"

Historical roots in the “processes and proficiencies" of, for eg.,

- NCTM Process Standards, 2000
- National Research Council’s Adding It Up Report, 2001
- National Math Advisory Panel Report, 2008

Goal and Overview of Session

Goal: To make sense of and articulate the practices in context of teacher mathematical investigation

- A mathematical activity: The Ruler Problem
- Small group discussion of practices
- Practices #1,2,6,7, and 8: key points and examples
- More ways to make sense of practices

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What do these Mathematical Practices mean...

to Teachers?

Are they Obvious?

- Are these just the usual “problem-solving" steps?
- Does “tools" in #5 just mean things like protractors and calculators?
- Does “precision" in #6 mean ‘enough decimal places’?

What’s the difference between #7 and #8?

- #7: Look for and make use of structure.
- #8: Look for and express regularity in repeated reasoning

What do they look like *in practice*?

What do Mathematical Practices mean...

to Mathematicians?

- Not explicitly taught (no ‘Math Research Methods’ course)
- Un-codified, highly individual, not limited to a single list
- Second nature or learned habits of mind?

How do those of us trained as mathematicians learn ‘mathematical practices’?

Learning Mathematical Practices

- Through **hands-on learning of unfamiliar mathematics** (reading papers, doing research)
- Through **collaboration/apprenticeship** with advisors, peers and colleagues
- **Over time** (K12, undergrad, grad, postgrad)

How can this inform *teachers* making sense of the mathematical practices?

Making Sense of Mathematical Practices

Give teachers experiences where they

- authentically use the practices

  low-threshold, but unfamiliar math context

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Making Sense of Mathematical Practices

Give teachers experiences where they

- authentically use the practices

  low-threshold, but unfamiliar math context

- explicitly reflect upon the practices

  articulate, discuss, debate them with others

Ruler Problem

Suppose you have 12-cm-long unmarked straight edge. What is the minimum number of marks you need in order to be able to *directly measure* all lengths 1 cm, 2 cm, ... up to 12 cm? (By *directly measure*, we mean the length is represented as the difference of two marks on the ruler.)

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Small group discussion

For your pair of practices, (1,5) (2,4) (3, 6) or (7,8), look for

- key points from the extended descriptions
- specific examples of these practices in the sample teacher work
- specific examples of these practices in your work on ruler problem

Other things to look for:

- *Which other practices arose in your work or the samples?*
- *How might you articulate these practices with middle school students?*

Strategy and Approach

- **1: Make sense of problems and persevere in solving them.**
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1: Make sense of problems and persevere in solving them.

Key points from the description:

- analyze givens, constraints, goals, relationships
- make conjectures about solution form and meaning
- plan a solution pathway
- try special cases/analogous problems
- monitor progress/change course if need be
- ask “Does this make sense?"
- ...

1: Make sense of problems and persevere in solving them.

Some Eg from Teacher Work: Cory and Paul

- plan a solution pathway
  generating examples and non-examples
  looking for what successful examples have in common
- make conjectures about solution form and meaning
  making conjectures about what you need for a "successful" ruler

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- monitor progress/change course if need be
  changing focus from trying eg. of rulers to number of marks
- *maintaining sight of goal
  realizing that the results from the chart helps get the final solution

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5: Use appropriate tools strategically.

From the description:

- consider available tools, including technology
- know how to use tools effectively
- know the limitations of certain tools and how to detect errors

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An Eg:

Cory using graph paper to visualize rulers

Representation and Modeling

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2: Reason abstractly and quantitatively.

From the description:

- make sense of quantities and relationships
- use properties of operations and objects flexibly
- decontextualize: abstract the situation, use symbols
- contextualize: refer back to what symbols represent/mean
- ...

2: Reason abstractly and quantitatively.

Some Eg from Teacher Work: Alice and Bob

- make sense of quantities and relationships
  trying to relate # marks to # lengths they could measure.
- decontextualize: abstract the situation, use symbols
  using symbols $A, B, C, \ldots$ to represent the marked positions
- contextualize: refer back to what symbols represent/mean
  trying specific values to think about different lengths, eg: if $C = 6$, then $12 - C = 6$ too

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- contextualize: refer back to what symbols represent/mean
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2: Reason abstractly and quantitatively.

Some Eg from Teacher Work: Eve

- decontextualize: abstract the situation, use symbols
  using x, y, z, w variables to represent the differences between marks
- contextualize: refer back to what symbols represent/mean
  realizing only sums of consecutive variables represented measurable lengths: $x + y$ but not $x + z$

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4: Model with mathematics.

From the description:

- apply math to everyday situations
- make assumptions and approximations to simplify situation
- identify important quantities
- analyze relationships mathematically
- evaluate model in light of situation
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An Eg:

Eve modeling the 'real-world' object (the ruler) as a sequence of variables which represent distance between marks

Communication and Reasoning

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3: Construct viable arguments and critique the reasoning of others.

From the description:

- understand and **use previous results**, assumptions, definitions
- make conjectures and explore them logically
- analyze situation using cases and counterexamples
- justify conclusions and evaluate others' reasoning
- ...

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An Eg:

Eve using Alice and Bob's results about 3 marks and 10 lengths to conclude that x, y, z, w *must be different values*

6: Attend to precision.

From the description:

- communicate precisely, give careful explanations
- use clear definitions
- use symbols appropriately
- be careful with units and axes
- calculate accurately and efficiently
- ...

6: Attend to precision .

Some Eg. from Teacher Work:

- communicate precisely, give careful explanations
  Cory & Eve being clear how their lists of numbers corresponded to markings on rulers
- use symbols appropriately
  Alice & Bob being clear about what $A, B, C, \ldots$ represented
  Eve being clear about what her variables represented ($x =$ distance to first mark, etc.)
- calculate accurately and efficiently
  Cory calculating all measurable lengths in an organized way

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6: Attend to precision orally and in written work.

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Structure and Generalization

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- **7: Look for and make use of structure.**
- **8: Look for and express regularity in repeated reasoning.**

7: Look for and make use of structure.

From the description:

- look for pattern/structure
- step back for an overview/shift perspective
- see something as a whole or as combination of parts
- *using **familiar/known structures** to see something in a different way

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7: Look for and make use of structure.

Some Eg. from Teacher Work:

- look for pattern/structure
  Paul: Noticing the symmetry in examples ('flips').
  Cory: Seeing that 1 or 11 always appears in successful ruler.
- *using **familiar/known structures** to see something in a different way
- see something as a whole or as combination of parts
  Alice recognizing triangular numbers and rewriting 3 = 1 + 2, 6 = 1 + 2 + 3, etc to explain why the triangular numbers appear

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8: Look for and express regularity in repeated reasoning.

From the description:

- notice repeated calculations ***and reasonings**
- look for general methods and shortcuts
- maintain oversight of process while looking at details
- evaluate reasonableness of results
- *finding **new structures/methods**, generalizing

8: Look for and express regularity in repeated reasoning.

Some Eg. from Teacher Work:

- look for general methods and shortcuts
  Bob generalizing pattern to find a formula for # lengths for a given # of marks on any size ruler

- notice repeated calculations (and reasonings)
  Paul realizing that Cory's reasoning about 1 and 11 extends to marks at 2, 10, or 11 are needed
  Eve generalizing her conclusions for $y = 2$ to other values of y (rather than checking all combinations of 1,2,3,4)

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Grouping Practices: Take One

- **Strategy and Approach**
  - 1: Make sense of problems and persevere in solving them.
  - 5: Use appropriate tools strategically.
- **Representation and Modeling**
  - 2: Reason abstractly and quantitatively.
  - 4: Model with mathematics.
- **Communication and Reasoning**
  - 3: Construct viable arguments and critique the reasoning of others.
  - 6: Attend to precision.
- **Structure and Generalization**
  - 7: Look for and make use of structure.
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Grouping Practices: Take Two

- **Overarch the Whole Process**
  - 1: Make sense of problems and persevere in solving them.
  - 6: Attend to precision.
- **Reasoning and Explaining**
  - 2: Reason abstractly and quantitatively.
  - 3: Construct viable arguments and critique the reasoning of others.
- **Modeling and Using Tools**
  - 4: Model with mathematics.
  - 5: Use appropriate tools strategically.
- **Structure and Generalization**
  - 7: Look for and make use of structure.
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Source: McCallum, 2011

Making Sense of the Practices

- **Grouping Practices:** Do some combine more naturally?
- **Looking Out for Practices:** How do they arise in math tasks/problems? in everyday situations?
- **Articulating Practices:** How would students say them? How to prompt students to use them?

Articulating Practices: An Eg.

2: Reason abstractly and quantitatively.

- decontextualize: abstract the situation, use symbols
  “zoom out. what’s the bigger picture? can you use a variable?"
- contextualize: refer back to what symbols represent/mean
  “zoom in. try an example with specific numbers or figures.

7: Look for and make use of structure:
“do you see a sheep in wolves clothing? a familiar structure beneath a complicated-looking surface? "

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- **Articulating Practices:** How would students say them? How to prompt them in students?
- **Connecting Practices:**
  - to NCTM processes
  - to NRC “Adding It Up" proficiencies
  - to content standards (“understand")

The NCTM Processes

- **Problem-solving**
  1: Make sense of problems and persevere in solving them.
- **Reasoning and Proof**
  2: Reason abstractly and quantitatively.
  3: Construct viable arguments and critique the reasoning of others.
  8: Look for and express regularity in repeated reasoning.
- **Communication**
  3: Construct viable arguments and critique the reasoning of others.
  6: Attend to precision.
  7: Look for and make use of structure.
- **Connections**
  5: Use appropriate tools strategically.
- **Representation**
  5: Use appropriate tools strategically.
  7: Look for and make use of structure.

Source: UAz video “Introduction to Practices #2" on Vimeo

The NRC “Adding It Up" Strands of Mathematical Proficiency

- **conceptual understanding**
  comprehension of mathematical concepts, operations, and relations
- **procedural fluency**
  skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- **strategic competence**
  ability to formulate, represent, and solve mathematical problems
- **adaptive reasoning**
  capacity for logical thought, reflection, explanation, and justification
- **productive disposition**
  habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy

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  habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy #1, 4: Make sense and persevere; Model with mathematics.

The Mathematical Practices are ...

- not new - have roots in previous work
- not a checklist of discrete things to do - highly interrelated
- not “eight commandments" - require interpretation in diverse contexts
- not obvious - require explicit articulation (we need to say what we’re doing)

The Mathematical Practices are ...

- basic skills teachers and students can develop and strengthen
- all about sense-making
- evident in the process of solving problems and learning math (not only/always in answers)
- real mathematics, not just a means to math

The Practices as "Powers"

"It has been well said that the highest aim in education
is analogous to the highest aim in mathematics, namely,
not to obtain results but powers,
not particular solutions, but
the means by which endless solutions may be wrought."

George Eliot, 1885

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Thank you!

Some Resources

- Common Core State Standards `www.corestandards.org`
- Focus on Math Math-Science Partnership `www.focusonmath.org`
- "Standards for Mathematical Practice: Building Awareness and Ideas for Implementation", Jim Mirabelli, Mathematics Assessment Specialist, Indiana Dept of Ed

  `media.doe.in.gov/WebEx/assessment/2012-01-26-TheStandardsforMathPractice/index.html`
- "Intro to Standards Math Practices", Institute for Math and Eduction, University of Arizona & Tucson Unified School District

  `vimeo.com/album/1750715/video/26977636`
- "Learning to Teach the Common Core", Deborah Loewenberg Ball, MSPnet Academy `hub.mspnet.org/index.cfm/webinars/webinar_info?id=16`

Thoughts? Questions? Please contact me at `jbelding@math.harvard.edu`