The Ruler Problem

Suppose you have 12-cm-long unmarked straight edge. What is the minimum number of marks you need in order to be able to directly measure all lengths 1 cm, 2 cm, ... up to 12 cm? (*By 'directly measure', we mean the length is represented as the difference of two marks on the ruler.*)

Small Group Activity:
Unpacking the Mathematical Practices

For your pair of practices, (1,5) (2,4) (3, 6) or (7,8), look for

- key points from the **extended descriptions**[1]
- specific examples of these practices in the **sample teacher work**[2]
- specific examples of these practices in your work on ruler problem

Be ready to share your findings within your small group.

The Standards for Mathematical Practice

1: Make sense of problems and persevere in solving them.

5: Use appropriate tools strategically.

2: Reason abstractly and quantitatively.

4: Model with mathematics.

3: Construct viable arguments and critique the reasoning of others.

6: Attend to precision.

7: Look for and make use of structure.

8: Look for and express regularity in repeated reasoning.

Other things to look for:

- *Which other practices arose in your work or the samples?*
- *How might you articulate these practices with middle school students?*

[1]These are from the Common Core State Standards for Mathematics document.

[2]The samples on pages 5-6 are adapted from actual work on the ruler problem by a Focus on Math study group of middle and high school teachers. The group work happened over the period of two 2-hour sessions. Here I've chosen snapshots of different teachers' work which show the evolution of the whole group's approach and solution to the problem. What mathematical practices do you see in play in this work?

Sample Teacher Work

Cory and Paul's Work

Cory and Paul began by trying different markings, recording their examples by noting the position of the marks, and looking for what the examples have in common. They came up with many "successful" examples (by "successful", they meant they could directly measure all lengths):

1, 5, 7, 9, 10   1, 3, 7, 11   1, 5, 9, 11   1, 4, 7, 10   2, 5, 8, 11   1, 4, 5, 10   1, 7 , 8, 10 . . .

They also came up with many examples that were not "successful":

1, 6, 8, 11 (can't measure 9) 1, 6, 9, 11 (can't measure 4, 7, 8) 1, 4, 7 (can't measure 2, 9, 10) 1, 5, 10 (can't measure 3, 6, 8) . . .

Cory used graph paper to make his examples, representing the ruler with 12 consecutive blocks and then looking for all lengths he could measure between marks by counting blocks. He eventually came up with a quicker way to check just using the numbers of marked positions:

Cory: *To make sure I'm not missing any measurable lengths, I check the difference between every pair of marks. I also like to think of 0 and 12 as marks when deciding which lengths can be measured because each position # is a measurable length and so are the numbers (12 - position #).*

From their examples, Cory and Paul concluded that the answer was less than or equal to 4, since they had examples of 4 marks which were successful (though not just any 4 marks would work). They also made the following observations, as they continued to test examples with 3 marks.

Paul: *You can 'flip' any marking to get a new one: for example, 1, 4, 7, 10 becomes 2, 5, 8, 11 by subtracting each position from 12:* $12-1=11, 12-4=8, 12-7=5, 12-10=2$. *If the original marking is successful, so is the new one, and vice-versa. So from our previous work, 2, 3, 5, 7, 11, and 2, 7, 8, 11 and 2, 4, 5, 11 are more successful examples. Also, for example, 2, 7, 11 is not successful because 1, 5, 10 is not.*

Cory: *All successful markings have either 1, 11 or both as marked positions. This is because you need either 1 or 11 to be able to measure 11. This means we only have to look at markings of this type.*

Paul: *Building off Cory's observation, I think that you need either 2, 10, or 11 in order to measure 10 (depending on if you've chosen 1 or 11 already).*

Alice and Bob's Work

Alice and Bob also created different examples and conjectured that you need 4 marks to make a successful example. To test this, they decided to look at what happens when using less than 4 marks. They labeled the ends with A and B (putting A at position 0 and B at position 12) and labeled remaining marks with $C, D, \ldots$ etc.

Alice: *If no other marks are added, then the only length we can measure is* $12 = B - A$. *If we use one mark (at position C), we get* $B-C$ *and* $C-A$ *too, for a total of 3. Putting mark* D *on creates* $B-D, D-C, D-A$, *for a total of 6.*

Bob: *But the lengths are not necessarily different! For example, if* $C = 6$, *then* C *and* $B-C$ *both are 6.*

They recorded their counts in a chart.

# marks	# marks + 2 (for the ends)	# lengths we can measure (not necessarily different)
0	2	1
1	3	3
2	4	6
3	5	...

Alice: *I notice the triangular numbers in the third column. I think this is because adding a new mark to* m *marks* *should create $m+2$ other lengths: the distances from the new mark to the previous m marks and the 2 ends. Using this, we can fill in the last line of the chart with $10 = 1+2+3+4$.*

# marks	# marks + 2 (for the ends)	# lengths we can measure (not necessarily different)	breakdown by number of new lengths we can measure for each new mark
0	2	1	
1	3	3	= 1 + 2
2	4	6	= 1 + 2 + 3
3	5	10	= 1 +2 + 3 + 4

Alice: *The fact that the lengths might not be different is very important: this number of lengths is the greatest number of different lengths we can hope to measure. Since 3 marks can measure 10 lengths, this means on a 12-ruler, we could never measure all 12 lengths with only 3 marks. So at least 4 marks are needed, like we thought!*

Bob: *We can make a formula another way too for the # lengths: it's just $(m+2)(m+1)/2$ since to count for all the differences you choose one mark from the $m+2$ marks, another from the remaining $m+1$. You divide by 2 because it doesn't matter the order you pick the two marks.*

This means the relationship between number of marks and lengths we can measure works for any size ruler, not just size 12. So for instance, if we have a length 10 ruler and we want measure all 10 lengths, we need at least 3 marks (because 2 marks can measure at most 6 different lengths). I wonder if we can do it with just 3 marks!

Eve's Work

To see if she could answer Bob's question, Eve tried lots of examples with 3 marks on a length 10 ruler. She recorded examples by noting the difference between the marks. For example, 1, 2, 4, 1 means the ruler has marks at positions 1, 3, 7, and 8. She decided to call the four consecutive differences x, y, z, w and to look for what values of these would make $x+y+z+w = 10$ and still make a successful example. She realized the lengths that could be measured were sums of consecutive variables: $x+y, y+z, z+w, x+y+z, y+z+w$, but not something like $x+z$.

Eve: *Because you can make only 10 lengths with 3 marks (by Alice and Bob's argument), I know we can't have any repeat lengths when measuring. This means we need x, y, z, w to all be different values. By trial and error, this means x, y, z, w have to be the values $1, 2, 3, 4$ in some order (because if we use numbers 5 or greater than the other three values can't all be different).*

Eve then tried to assign the values $1, 2, 3, 4$ to x, y, z, w in different orders to get successful rulers. To test if a ruler was successful, she looked at the values of the sums $x+y, y+z, z+w, x+y+z, y+z+w$ to see if she could get 5, 6, 7, 8 and 9.

Eve: *Using Cory's observation about 1 or 11 on a 12-ruler, I know I need a mark at 1 or 9 to measure length 9 on a 10-ruler. Since I can always think about the flip, I'll let $x = 1$ and try different values for y, z, w.*

She recorded her work in a chart, looking out for "repeats", values which were measured twice.

$y =$	$w =$	$z =$	repeats
2	3	4	3 is measured twice, as w and as $x+y$
2	4	3	3 is still measured twice, as w and as $x+y$

Eve: *So I can't have $y = 2$ because no matter what w, z are, 3 is still measured twice. I can use the same argument for $y = 3$ and $y = 4$, so it's impossible to have a successful example of a 10-ruler with only 3 marks!*

$y =$	$w =$	$z =$	repeats	general conclusion
2	3	4	3 is measured twice, as w and as $x+y$	
2	4	3	3 is still measured twice, as w and as $x+y$	so can't have $y = 2$
3	2	4	4 is measured twice, as z and as $x+y$	can't have $y = 3$
4	2	3	5 is measured twice, as $x+y$ and as $w+z$	can't have $y = 4$

Mathematics | Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3 Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5 Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6 Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7 Look for and make use of structure.
Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 × 8 equals the well remembered 7 × 5 + 7 × 3, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2 × 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 – 3(x – y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8 Look for and express regularity in repeated reasoning.
Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y – 2)/(x – 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x – 1)(x + 1)$, $(x – 1)(x^2 + x + 1)$, and $(x – 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content
The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential “points of intersection” between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.